My rant about stress, strain, and elasticity in planetary sciences

by Peter Bird, UCLA, November 2018

Most textbooks about tectonics (and almost all textbooks about seismology) begin their continuum-mechanics section with a simple equation describing elasticity in terms of stress tensor $\tilde{\sigma}$, strain tensor $\tilde{\varepsilon}$, and a fourth-rank tensor that relates them. After the introduction of a coordinate system, so that stress and strain can each be represented by components (σ_{ij} and ε_{kl} , respectively), then we will either see a relationship in terms of stiffness (moduli) $\tilde{K}$:

$$\sigma_{ij} = K_{ijkl}\,\varepsilon_{kl} \tag{1}$$

or the inverse relation, in terms of a fourth-rank compliance tensor $\tilde{C}$:

$$\varepsilon_{kl} = C_{klij}\,\sigma_{ij} \tag{2}$$

These two equations are usually presented as generally-accepted and noncontroversial. However, are they really generalizations from millions of precise scientific measurements? Or, are they merely assertions of religious faith?

Measuring strain is very difficult, and rarely done. Assume you can find a perfect single crystal (with no cracks, inclusions, shear bands, twin boundaries, dislocations, vacancies, interstitials, or impurities) and that it has a convenient rectangular shape. In the laboratory, you can force this crystal to strain, and then measure the strain (relative displacements of atoms) through changes in its unit-cell dimensions by the X-ray diffraction method. (It is best to do this measurement near absolute zero to minimize thermal broadening of the X-ray diffraction peaks.) The precision will not be very good unless great care and expense is taken to optimize the X-ray optics and thus prevent geometric broadening of X-ray diffraction peaks. Alternatively, you could measure the average strain across the whole crystal using laser interferometry; this would give better precision.

Alleged “measurements of strain” using glued-on “strain gauges” are actually just measurements of the ***changes*** in strain since the application of the gauge(s). Also, such measurements are limited to accessible free surfaces.

Measuring strain in rocks that are buried underground is virtually impossible. Measuring strains underground in the geologic past is doubly impossible.

{Measuring stress is also very difficult, and rarely done. Techniques such as overcoring, hydrofracturing, and flat-jacking give rough estimates (with ~10% precision) but they are invasive measurements because they require cutting into the rock (causing permanent damage), replacing a part of the rock, and then trying to return the system to something like its initial state (ignoring hysteresis and new cracks). One can do somewhat better when estimating average values of one traction component on a surface; for example, if you place a cube of iron on top of the perfect rectangular crystal mentioned above, you can compute the average value (across its upper surface) of the vertical tractions exchanged between iron and crystal by knowing the dimensions of the iron block, the mass and radius of the Earth, and the gravitational constant. But obviously this is a model-derived estimate rather than a measurement. You would have to do theoretical or numerical modeling of the system to estimate how the vertical traction is nonuniformly distributed across the crystal face, and this work would involve many assumptions and approximations.}

Another complication for rocks contained in the lithosphere of a terrestrial planet is that they almost always show evidence for large ***non-elastic strains***. (One possible counter-example would be a perfect crystal of halite just formed at the edge of a tide pool.) Crystals in an igneous rock have undergone different amounts of (usually anisotropic) thermal contraction; thus they will either be cracked (and then weathered), or else locked in complex self-equilibrating loops of "pre-stress" which will cause non-elastic deformations by solution-transfer creep and diffusion creep and dislocation creep over time. Sedimentary rocks show evidence of past compaction by a combination of frictional plasticity and solution-transfer creep. (The latter strain mechanism was sometimes formerly known as "pressure-solution".) Metamorphic rocks may have undergone non-elastic strains of order unity, or much more. At a larger scale, almost all outcrops, roadcuts, and quarry faces show additional inhomogeneous non-elastic deformation expressed as joints and/or faults.

Given the practical impossibility of measuring total strain in planetary sciences, and the extremely high probability of non-elastic strain contributions in the history of most rocks, a far more honest statement of elasticity would be either:

$$\Delta\varepsilon_{kl}^{\text{elastic}} = C_{klij}\,\Delta\sigma_{ij} \tag{3}$$

or

$$\Delta\sigma_{ij} = K_{ijkl}\,\Delta\varepsilon_{kl}^{\text{elastic}} \tag{4}$$

where Δ indicates the ***change*** over some well-defined interval of (laboratory or geologic) time during which there was no change in temperature. Note that the total strain change tensor $\Delta\tilde{\varepsilon}$ (which one might estimate from geodesy) often involves both elastic and non-elastic parts, and that additional equations will be necessary to describe any non-elastic strain changes.

These improved (incremental and isothermal) elastic "laws" are perfectly adequate for deriving all the classical results of seismology. In the first few chapters, attention could be focused on equation (3) or (4) for cases which have only elastic strain changes, and the existence of waves could be derived. In later chapters an equation for changes in viscous strain could be added, leading to the finding of seismic attenuation (in the sense of heat-generation, not in the sense of geometric spreading).

Naturally, ***the total strain change over any given time interval is the sum of the strain changes due to the many simultaneous mechanisms*** of elasticity, thermal expansion, phase changes, cracking, frictional plasticity, dislocation creep, solution transfer creep, Nabarro-Herring creep, *etc.*

However, ***the change in stress is not a sum***; there is only one (unique) stress tensor at any point and time. If one insists on talking about "frictional stress" or "elastic stress" (*etc.*) which are derived from some single-mechanism flow-law found in the literature, then it must be understood that the single true stress is a tensor whose *deviator* is no larger than the *deviator* of any of the single-mechanism theoretical "stresses" that provide alternative upper limits.

An important consequence of these arguments is that ***tectonic modeling can rarely be based on elastic equations***. If equations (1) and (2) are used, the work is philosophical rather than scientific, because it refers to a quantity (strain) that is not measurable in planetary interiors. Even if improved equations (3) and/or (4) are used, ***it will be necessary to verify that elastic strain is the primary cause of strain changes*** during the modeled process, by successfully returning the system to its initial state. This limits elastic models to short time-spans, to the shallow parts of the crust (where there is less thermally-activated creep), and to cases of low deviatoric stress and/or reductions in deviatoric stress (so that additional strain changes due to frictional plasticity, cracking, and/or phase changes are less likely).

Such incrementally-elastic models, where permissible, can never reach any conclusions about the absolute level of stress in the lithosphere, because equations (3, 4) make it very clear that ***they only predict changes in stress***. If the initial values are unknown, the final values will also be unknown.

The modeling of long geologic time-spans, the full thickness of the lithosphere, and deviatoric stresses large enough to cause permanent geologic structures (e.g., faults, folds, and unconformities) must be based on the mathematics of stress-equilibrium (i.e., momentum-conservation) combined with mathematics of the relevant permanent-strain mechanisms, such as frictional plasticity at low temperature, and dislocation-creep at high temperature.