Appendix to the Appendices

by Peter Bird, UCLA, 2017.04.24

Abstract: In the **Appendix** to *Bird* [1998, *Tectonics*], which gave the algorithm for program *Restore* used in the reconstruction of the Rocky Mountains, equation (12) asserts that the continuum contribution to the covariance of strain rates in any faulted finite element is

$$\tilde{V} = \mu^2 \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & 4/3 \end{bmatrix} \qquad \text{[AA1]}$$

where it should be understood that the 3 independent components of the 2x2 strain rate tensor in the horizontal plane ($\dot{\varepsilon}_{\theta\theta} = \dot{\varepsilon}_{NS}, \dot{\varepsilon}_{\theta\phi} = \dot{\varepsilon}_{SE}, \dot{\varepsilon}_{\phi\phi} = \dot{\varepsilon}_{EW}$) have been arranged into a one-subscript vector ($\dot{\varepsilon}_q;\ q = 1, 2, 3$), permitting us to write the covariance of strain-rates as a 3x3 matrix instead of a 2x2x2x2 hyper-tensor. The same claim is repeated in equation (19) of **Appendix I: Algorithm of NeoKinema** by *Bird* [2005] which was published in connection with several papers using the *NeoKinema* algorithm. The reason for choosing these particular coefficients are not obvious. This short white-paper will show that the coefficients 4/3 and -2/3 are not unique, but are probably the most reasonable choices.

Background: A basic strategic choice in the design of programs *Restore* and *NeoKinema* was that, in the absence of any local data (faults with offsets or slip rates, GPS velocities, stress directions, *etc.*), the lithosphere should behave as a thin, isotropic, laterally-homogeneous, incompressible viscous shell in plane stress. In the case of isotropic Newtonian viscosity with coefficient ν , and of a traction-free upper boundary condition with $\tau_{r\theta} = \tau_{r\phi} = 0$, the viscous dissipation rate (per unit volume) is

$$\dot{Q} = \tau_{ij}\dot{\varepsilon}_{ij} = 4\nu\left[\dot{\varepsilon}_{\theta\theta}^2 + \dot{\varepsilon}_{\theta\theta}\dot{\varepsilon}_{\phi\phi} + \dot{\varepsilon}_{\phi\phi}^2 + \dot{\varepsilon}_{\theta\phi}^2\right] = 4\nu\left[\dot{\varepsilon}_1^2 + \dot{\varepsilon}_1\dot{\varepsilon}_3 + \dot{\varepsilon}_3^2 + \dot{\varepsilon}_2^2\right] . \qquad \text{[AA2]}$$

Consequently, the objective functions for both *Restore* and *NeoKinema* include the term

$$\frac{\dot{\varepsilon}_{\theta\theta}^2 + \dot{\varepsilon}_{\theta\theta}\dot{\varepsilon}_{\phi\phi} + \dot{\varepsilon}_{\phi\phi}^2 + \dot{\varepsilon}_{\theta\phi}^2}{\mu^2} . \qquad \text{[AA3]}$$

In the absence of other terms, the minimization of this quantity will yield the behavior of a uniform, isotropic, incompressible viscous sheet. Unless driven by velocity boundary conditions (or distant data), its strain rates will be uniformly zero.

Strain-Rate Perturbation Vectors: Given this context, it is reasonable to choose strain-rate perturbation vectors (P_i^k , where $q = 1, 2, 3$ for the strain-rate component, and $k = 1, 2, 3$ for the different perturbations) which are normalized so that

$$\dot{\varepsilon}_{\theta\theta}^2 + \dot{\varepsilon}_{\theta\theta}\dot{\varepsilon}_{\phi\phi} + \dot{\varepsilon}_{\phi\phi}^2 + \dot{\varepsilon}_{\theta\phi}^2 = \mu^2 . \qquad \text{[AA4]}$$

Because there are 3 independent strain rates, there should be 3 linearly-independent strain-rate perturbation vectors ($k = 1, 2, 3$). However, there are more than 3 "natural" choices:

(A) $\dot{\varepsilon}_{\theta\phi} = \mu$; other 2 are zero. (Note implied $\dot{\varepsilon}_{rr} = 0$.)

(B) $\dot{\varepsilon}_{\theta\theta} = \mu$; other 2 are zero. (Note implied $\dot{\varepsilon}_{rr} = -\mu$.)

(C) $\dot{\varepsilon}_{\phi\phi} = \mu$; other 2 are zero. (Note implied $\dot{\varepsilon}_{rr} = -\mu$.)

(D) $\dot{\varepsilon}_{\theta\theta} = \mu$; $\dot{\varepsilon}_{\phi\phi} = -\mu$; $\dot{\varepsilon}_{\theta\phi} = 0$. (Note implied $\dot{\varepsilon}_{rr} = 0$.)

(E) $\dot{\varepsilon}_{\theta\theta} = \frac{\mu}{\sqrt{3}}$; $\dot{\varepsilon}_{\phi\phi} = \frac{\mu}{\sqrt{3}}$; $\dot{\varepsilon}_{\theta\phi} = 0$. (Note implied $\dot{\varepsilon}_{rr} = \frac{-2\mu}{\sqrt{3}}$.)

(F) $\dot{\varepsilon}_{\theta\theta} = \mu\sqrt{\frac{4}{3}}$; $\dot{\varepsilon}_{\phi\phi} = -\frac{\mu}{2}\sqrt{\frac{4}{3}}$; $\dot{\varepsilon}_{\theta\phi} = 0$. (Note implied $\dot{\varepsilon}_{rr} = -\frac{\mu}{2}\sqrt{\frac{4}{3}}$.)

(G) $\dot{\varepsilon}_{\phi\phi} = \mu\sqrt{\frac{4}{3}}$; $\dot{\varepsilon}_{\theta\theta} = -\frac{\mu}{2}\sqrt{\frac{4}{3}}$; $\dot{\varepsilon}_{\theta\phi} = 0$. (Note implied $\dot{\varepsilon}_{rr} = -\frac{\mu}{2}\sqrt{\frac{4}{3}}$.)

Some obvious permutations that preserve linear-independence are:

I: (A) & (B) & (C)

II: (A) & (D) & (E)

III: (A) & (F) & (G)

<u>Covariance matrices</u>: As in the analogous equation (2) of *Bird & Carafa* [2016, *J. Geophys. Res.*]:

$$V_{ij} = \sum_{k=1}^{3} P_i^k \, P_j^k \quad , \qquad \text{[AA5]}$$

we find 3 different estimates for $\tilde{V}$ by using these different permutations:

Permutation I: $\tilde{V} = \mu^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mu^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Permutation II: $\tilde{V} = \mu^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} + \mu^2 \begin{bmatrix} 1/3 & 0 & 1/3 \\ 0 & 0 & 0 \\ 1/3 & 0 & 1/3 \end{bmatrix} = \mu^2 \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 1 & 0 \\ -2/3 & 0 & 4/3 \end{bmatrix}$.

Permutation III:

$$\tilde{V} = \mu^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu^2 \begin{bmatrix} 4/3 & 0 & -2/3 \\ 0 & 0 & 0 \\ -2/3 & 0 & 1/3 \end{bmatrix} + \mu^2 \begin{bmatrix} 1/3 & 0 & -2/3 \\ 0 & 0 & 0 \\ -2/3 & 0 & 4/3 \end{bmatrix} = \mu^2 \begin{bmatrix} 5/3 & 0 & -4/3 \\ 0 & 1 & 0 \\ -4/3 & 0 & 5/3 \end{bmatrix}.$$

Selection: An interesting coincidence is that the covariance matrix in Permutation II is the average of the covariance matrices from Permutation I and Permutation III. Because it is both the central and the mean estimate, I adopted Permutation II as giving the most reasonable estimate of the continuum contribution to the covariance of strain-rates, for use in both the *Restore* and *NeoKinema* algorithms. However, it is admittedly not unique.