Isotropic_viscous_lithosphere.docx & .pdf

by Peter Bird, 2019.05.13

showing derivation of the viscous strain-rate term in the NeoKinema objective function.

The general relation between strain-rate and deviatoric stress in viscous materials involves a fourth-rank tensor of viscosity coefficients. However, we will probably never have sufficient data to constrain so many coefficients. A popular and reasonable approximation is to begin with an assumption of isotropic viscosity. Here we also assume that the viscous flow-law is also linear or "Newtonian" (as opposed to "power-law") with respect to the magnitudes of strain-rates.

According to Gerald Schubert, Donald L. Turcotte, & Peter Olson, 2001,
Mantle Convection in the Earth and Planets,
"viscous dissipation" (mechanical energy converted to heat, per unit volume, per unit time) is:

$$\Phi = \tau_{ij} \frac{\partial u_i}{\partial x_j} \quad (6.9.3)$$

where τ_{ij} are stress components, u_i are velocity components, and x_j are Cartesian coordinates.

We also use their formula for stress in an isotropic, linear-viscous fluid,

$$\tau_{ij} = 2Ve_{ij} + \left(k_{\mathrm{B}} - \frac{2}{3}V \right) e_{kk} \delta_{ij} \qquad (6.5.4)$$

where e_{ij} is a strain-rate component, δ_{ij} is a component of the Kronecker-delta matrix (identity matrix),
and V and k_{B} are shear- and bulk-viscosity parameters, respectively.

If viscous straining is the only kind of straining, and if volume is conserved, then $e_{kk} \equiv 0$, and the right-hand term of (6.5.4) can be dropped. Now,

$$\Phi = \frac{V}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2 \qquad (6.9.14)$$

which implies a sum over both *i* and *j*.

Let us rotate the coordinate axes to be parallel to the principal strain-rate axes.

Then, there are only 3 non-zero strain-rates (on the diagonal), and no rotation-rates,

and thus $\partial u_i / \partial x_j = 0$ for all cases where $i \neq j$.

Thus, we can rewrite (6.9.14) as

$$\Phi = \frac{V}{2}\left[\left(2e_{11}\right)^2 + \left(2e_{22}\right)^2 + \left(2e_{33}\right)^2\right] = 2V\left(e_{11}^2 + e_{22}^2 + e_{33}^2\right)$$

.

(Note that the ordering of the 3 axes is arbitrary, with no implication about relative magnitudes.)

Let $\hat{x}_3$ be the vertical direction. Since the lithosphere has a traction-free upper surface (except for atmospheric pressure and ocean pressure) the vertical direction is a principal-stress and principal-strain-rate axis. Then, using incompressibility again, replace

$e_{33} = -(e_{11} + e_{22})$.

Then, in terms of the 2 horizontal principal strain-rates,

$$\Phi = 2V\left(e_{11}^2 + e_{22}^2 + (e_{11} + e_{22})^2\right)$$

which is algebraically equivalent to:

$$\Phi = 2V\left(2e_{11}^2 + 2e_{11}e_{12} + 2e_{22}^2\right) = 4V\left(e_{11}^2 + e_{11}e_{22} + e_{22}^2\right)$$

Now, consider non-principal-axis horizontal coordinate directions (θ, ϕ),

with new axis direction $\hat{\theta}$ at an angle α (measured counterclockwise, in radians) from $\hat{x}_1$.
Using Mohr's circle for strain-rates,

define $c = \dfrac{e_{\theta\theta} + e_{\phi\phi}}{2}$ and $r = \sqrt{\left(\dfrac{e_{\theta\theta} - e_{\phi\phi}}{2}\right)^2 + e_{\theta\phi}^2}$, and then we

can express

$e_{11} = c - r$; $e_{22} = c + r$ (or the same with subscripts 1 & 2 reversed), and
therefore

$$\Phi = 4V\left((c-r)^2 + (c-r)(c+r) + (c+r)^2\right) =$$

$$4V\left(3c^2 + r^2\right) = 4V\left(e_{\theta\theta}^2 + e_{\theta\theta}e_{\phi\phi} + e_{\phi\phi}^2 + e_{\theta\phi}^2\right).$$

The quantity in the parentheses, in the right-hand form, is the quantity that is minimized in the objective function of **NeoKinema**. It is discussed specifically in equation (8) of the document "Appendix-Algorithm of NeoKinema".

Note that the ***kinematic*** F-E code **NeoKinema** replaces the viscosity coefficient ($4V$) of this ***dynamic*** derivation with a coefficient of $(1/\mu^2)$, where μ (or mu_ in Fortran code) is a scalar
characteristic strain-rate. This is reasonable because the stress-equilibrium equation
(and plate-driving mantle dynamics) lead to deviatoric stress fields which are laterally smooth;
therefore regions of low strain-rates are regions of high viscosity, and *vice versa*.
The other reason for this substitution is that Φ is in dimensional units (W m^{-3}),

but the objective function for **NeoKinema** requires terms that are dimensionless.