Earth and Planetary Science Letters, 45 (1979) 214-217

[2]

THE USE OF THE MINIMUM-DISSIPATION PRINCIPLE IN TECTONOPHYSICS

PETER BIRD

Department of Earth and Space Sciences, University of California, Los Angeles, CA 90024 (U.S.A.)

and

DAVID A. YUEN

Department of Physics, University of Toronto, Toronto, Ont. M5S 1A7 (Canada)

Received August 9, 1978
Revised version received March 29, 1979

The principle, that the rate of internal viscous dissipation is at a minimum, is incorrect when temperature and velocity fields are linked through temperature-dependent viscosity or density. This makes it inappropriate for the study of spreading ridge – transform fault systems or other plate-tectonic problems with large density stratification resulting from large temperature gradients. Corrections to the principle are noted for cases without heat advection but with boundary tractions or non-linear materials.

In this essay we caution against three types of unjustified application of the minimum-dissipation principle to problems of flow in the Earth. The first is the failure to include appropriate boundary terms when a part of the Earth is isolated for study. The second is improper scaling of internal dissipation terms when the material has a power-law rheology. Finally, we consider the breakdown of the principle when temperature and flow fields become linked – as they are in plate tectonics. This third problem is fundamental, because there is no other functional whose minimization (with respect to flow geometry) will solve these problems either. We conclude that laboratory modeling or direct solution of the governing differential equations are the only valid approaches.

These difficulties have been understood and mentioned in the applied mathematics literature for years, and indeed are readily apparent when one attempts to derive the minimum-dissipation principle from physical conservation equations. We restate them here because we see a possibility that the principle might come to be regarded as the equivalent of the second law of thermodynamics in the geophysical literature. Recently, many authors [1–6] have assumed this principle without detailed justification and applied it without observing some necessary restrictions or assumptions.

Allow us to define a functional which includes *half* of the viscous dissipation rate, *minus* the rates of work done by pressure, gravity, and boundary tractions:

$$\pi \equiv \iiint_v \left[\frac{1}{4}\eta \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i} \right) - P \frac{\partial}{\partial x_i}(V_i) - X_i V_i \right] \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z - \oint_{s_{\text{free}}} \left[\tau_i V_i \right] \mathrm{d}s \qquad (1)$$

Here and throughout the paper we use the summation convention. V_i represents the velocity vector, P is pressure, η is viscosity, and X_i is the body force/volume vector. The last term is the surface integral of the scalar product of velocity with the surface force/area vector, which can also be written as:

$$\oint_{s_{\text{free}}} \left[\tau_i V_i \right] \mathrm{d}s = \oint_{s_{\text{free}}} \left[V_i \sigma_{ij} \xi_j \right] \mathrm{d}s$$

in terms of stress tensor σ_{ij} and the vector ξ_j of direction cosines of the outward normal direction. The surface s_{free} is *only* that surface on which velocity is not specified. This means that if the problem includes no body forces and if the system is either rigidly constrained or free of external forces, we return to the internal-dissipation functional.

At a minimum of functional (1), it must be stationary:

$$\delta\pi = 0 = \frac{\partial\pi}{\partial V_i}\delta V_i + \frac{\partial\pi}{\partial\left(\partial V_i/\partial x_j\right)}\frac{\partial\left(\delta V_i\right)}{\partial x_j} + \frac{\partial\pi}{\partial\eta}\delta\eta + \frac{\partial\pi}{\partial P}\delta P + \frac{\partial\pi}{\partial X_i}\delta X_i + \frac{\partial\pi}{\partial\tau_i}\delta\tau_i \tag{2}$$

The great value of this potential to date has been in engineering problems in which the viscosity, density, and surface tractions are specified and not varied. That is,

$$\delta\eta = \delta X_i = \delta\tau_i = 0 \tag{3}$$

In that case we can solve (2) by requiring the remaining terms to vanish. Making use of Gauss' method of integration by parts and the commutativity of the operators δ and $\partial/\partial x_i$ [7], we find the Euler equations (which are the coefficients of the pressure and velocity variations). For pressure:

$$\frac{\partial}{\partial x_i}(V_i) = 0 \tag{4}$$

This is the usual Boussinesq or incompressible approximation of mass-conservation. (Although it is essential to take account of adiabatic compression effects for heat-conservation [8], this is entirely adequate to constrain the form of the flow.) The other Euler equations are those of momentum-conservation in the absence of significant acceleration:

$$X_i - \frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\right\} = 0; \quad i = 1,2,3 \tag{5}$$

Since the minimization of π is equivalent to (4) and (5), we see that the minimum-dissipation principle as defined here is correct for creeping flows with predetermined viscosity and density.

The first questionable step is the neglect of the surface integral in (1). Mathematically, it is required to cancel out the boundary term which arises when the velocity and pressure terms of (2) are integrated by parts in search of Euler equation (5). It can be dropped only if there is no surface with free velocity, or no traction on such surfaces. The authors of papers [2], [3], [5], and [6] have neglected this surface term, and in so doing have made an unstated assumption that the parts of their model domains which do not have velocity boundary conditions are in contact with perfect fluids. In view of the increasing evidence for whole-mantle convection with no perfect decoupling, this seems to be an assumption that requires some justification. The only way to avoid *some* approximation of the interaction of the different parts of the Earth is to extend the domain to the whole solid planet, and minimization principles should not be used to disguise this unpleasant fact.

When boundary (or gravity) forces are added to the problem in addition to velocity boundary conditions, it becomes important to multiply the local viscous dissipation in (1) by the correct constant. Note that when η is independent of V we minimize *one-half* of the dissipation rate, in order to get the correct constant coefficient of viscosity in (5). Similar caution is required when the viscosity depends on the rate-of-strain, as it usually does in Earth materials [9]. Bird [10] has shown that there is a more general functional which is valid for such variable viscosity. He states that when a power-law rheology is used:

$$\eta = a\left[\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\right]^b \tag{6}$$

that the new functional reduces to the ordinary dissipation functional. We wish to correct a second error by pointing out that in this case the $\partial\pi/\partial\eta$ term of (2) becomes non-zero and when integrated contributes additional factors of the stress gradient to the one obtained from the $\partial\pi/\partial(\partial V_i/\partial x_j)$ term. Therefore it is necessary to start with a corrected form of (1) which begins:

$$\pi \equiv \iiint_v \left[\frac{1}{4(b+1)}\eta\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) - \ldots\right.$$

in order to recover (5) from its minimization. Obviously this is equivalent to (1) for $b = 0$. In the case where different parts of the domain have different b values, b is considered a variable in the above volume integral. Paper [6] fails to take this into account in determining the minimum dissipation configuration of ridges and transforms of different power-law rheologies.

The most serious of the three errors, though, is to assume that (1) still implies (5) when temperature controls viscosity and density and is in turn controlled by the flow. This case of profoundly non-linear physics is dominant in plate tectonics, where the flow has a large Nusselt number and the materials have a temperature-dependent viscosity of the form [9]:

$$\eta = aT\left[\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)\right]^b \exp\left(\frac{c}{T}+\mathrm{d}P\right) \quad (7)$$

For example, most of the papers cited deal with mid-ocean spreading ridges, which are hot regions maintained by convective flow. A viscosity contrast of several orders of magnitude is caused by this thermal anomaly and is essential to the mechanics of the ridge [11, 12].

The approach of all authors to date has been to implicitly calculate the temperature field associated with a trial flow field by solving the steady-state heat-conservation equation:

$$\begin{aligned}&\nabla\bullet(K\vec{\nabla}T)+H+\alpha TV_i\frac{\partial P}{\partial x_i}-\rho C_p V_i\frac{\partial T}{\partial x_i}\\&+\frac{\eta}{2}\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)\left(\frac{\partial V_i}{\partial x_j}+\frac{\partial V_j}{\partial x_i}\right)=\rho C_p\frac{\partial T}{\partial t}=0\end{aligned} \quad (8)$$

where K is conductivity, T is Kelvin temperature, H is radioactive heat production, α is the thermal expansion coefficient, C_p is heat capacity, and t is time. Equation (8) may be represented by intuitive or approximate adjustments of viscosity, as in [1–3] and [6] where the thermally-weakened ridge center is assumed to vary its geometry as the trial flow field is varied.

The problem with this approach is that the viscosity and body force variations in (2) become non-zero. To consider only the former:

$$\begin{aligned}&\frac{\partial\pi}{\partial\eta}\delta\eta=\frac{\partial\pi}{\partial\eta}\left(\frac{\partial\eta}{\partial T}\delta T+\frac{\partial\eta}{\partial P}\delta P\right.\\&\left.+\frac{\partial\eta}{\partial(\partial V_i/\partial x_j)}\frac{\partial(\delta V_i)}{\partial x_j}\right)\end{aligned} \quad (9)$$

Again, consider only the first term of (9):

$$\begin{aligned}&\frac{\partial\pi}{\partial\eta}\frac{\partial\eta}{\partial T}\delta T=\frac{\partial\pi}{\partial\eta}\frac{\partial\eta}{\partial T}\left(\frac{\partial T}{\partial(\partial P/\partial x_i)}\frac{\partial(\delta P)}{\partial x_i}\right.\\&\left.+\frac{\partial T}{\partial V_i}\delta V_i+\frac{\partial T}{\partial(\partial V_i/\partial x_j)}\frac{\partial(\delta V_i)}{\partial x_i}\right)\end{aligned} \quad (10)$$

Assuming that one succeeds in minimizing (1) with auxiliary conditions (7) and (8), these many new terms in (2) will result in Euler equations which are different from the Stokes equation (5) and are non-physical. To put it another way: at a point in $(\vec{V},T)$ space where the Stokes equation (5) and the heat equation (8) are satisfied, the new terms in (2) will cause the variation of π to be non-zero, so the functional cannot have a minimum at that point. It is hard to predict in general what velocity field will give the minimum dissipation, but in general it will not be in static equilibrium. Thus the "stable" ridge-transform angles computed by the above authors have questionable validity.

Attempts have been made to find a new functional whose minimization will also imply (8) and thus solve the problem. They have not succeeded because the heat equation contains advective terms $\vec{V}\bullet\vec{\nabla}T$ and $\vec{V}\bullet\vec{\nabla}P$ which would require this new functional to have a non-symmetric Frechet differential. It can be shown [13, 14] that *no* functional exists whose global minimization yields such differential equations. This is why the Helmholtz minimization theorem holds only for slow flow, where inertial terms $\vec{V}\bullet\vec{\nabla}\vec{V}$ are neglected [15]. Finlayson shows that if one is *also* willing to impose the *adjoint* of the physically-motivated differential equations, then a functional can be found whose minimization yields them both. However, in the case of non-linear equations such as these, the adjoint condition is complex and frequently non-physical. In general, points in $(\vec{V},T)$ space which satisfy the physical differential equations will not satisfy the adjoint equations, and so cannot be located by the vanishing of the variation of these "extended functionals".

The only method now known for solving non-linear, non-self-adjoint partial differential equations by use of a functional is the "local potential" method, developed by Glansdorff and Prigogine [16], and applied by Lebon and Mathieu [17] and others. Its essence is the construction of a functional of both velocity *and* temperature and some small perturbations of each. This functional is written so that the physical differential equations are recovered by minimizing it with respect to the V and T variations *alone* (holding the trial solution fixed), and the subsidiary condition that the variations giving this minimum be zero. Although mathematically interesting, this type of functional is not very useful in choosing the best of many approximate solutions as the above-mentioned authors have done. The functional will not have a physical significance in convective problems like plate tectonics, so the comparison of functional values for competing solution estimates at zero perturbation means nothing.

In short, geophysicists would do well to give up the search for global solution methods for temperature-dependent flow problems. The minimum-dissipation principle is really only good for finding an upper bound to the dissipation in a very restricted class of problems (equation (3)). For the advective problems characterizing plate tectonics, an appropriately-scaled laboratory model will yield reliable conclusions *and* new insights (e.g. [18]). When appropriate materials do not exist, one must solve the governing differential equations directly by analytical or numerical means. Of these, the most flexible are the finite-difference and finite-element methods, which can in principle be applied to any differential equation. Glansdorff and Prigogine [16] have shown that their "local potential" method is equivalent to the self-consistent Galerkin method that underlies finite-element solutions. Gallagher et al. [19] present a comprehensive collection of these methods. Whatever course is chosen, a combination of stability analysis and physical insight must be used in interpreting the results, because of the possibilities of multiple solutions, oscillations, or instabilities which make the study of geology so interesting.

Acknowledgements

We thank F. Busse, R. Chin, R. Geller, and S. Stein for stimulating conversations. This research was supported by the Earth Sciences Division, National Science Foundation, under grants EAR-77-15198 and EAR-78-22753.

References

1 A.H. Lachenbruch and G.A. Thompson, Oceanic ridges and transform faults: their intersection angles and resistance to plate motions, Earth Planet. Sci. Lett. 15 (1972) 116.

2 H. Aoki, Stability of transform faults as inferred from viscous flow in the upper mantle, J. Phys. Earth 21 (1973) 97.

3 C. M. Froidevaux, Energy dissipation and geometric structure at spreading plate boundaries, Earth Planet. Sci. Lett. 20 (1973) 419.

4 P. Bird, Thermal and mechanical evolution of continental convergence zones: Zagros and Himalaya, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, Mass. (1976) 423 pp.

5 S. Stein, H.J. Melosh, and J.B. Minster, Ridge migration and asymmetric sea floor spreading, Earth Planet. Sci. Lett. 36 (1977) 51.

6 S. Stein, A model for the relation between spreading rate and oblique spreading, Earth Planet. Sci. Lett. 39 (1978) 313.

7 F. B. Hildebrand, Methods of Applied Mathematics (Prentice-Hall, Englewood Cliffs, New York, 1965) pp. 135-139.

8 D.C. Turcotte, A.T. Hsui, K.E. Torrance and G. Schubert, Influence of viscous dissipation on Benard convection, J. Fluid Mech. 64 (1974) 369.

9 J. Weertman and J.R. Weertman, High temperature creep of rock and mantle viscosity, Annu. Rev. Earth Planet. Sci. 3 (1975) 293.

10 R.B. Bird, New variational principle for incompressible non-Newtonian flows, Phys. Fluids 3 (1960) 539.

11 K. Fujita and N.H. Sleep, Membrane stresses near mid-ocean ridge-transform intersections, Tectonophysics 50 (1978) 207.

12 P. Tapponier and J. Franchetau, Necking of the lithosphere and the mechanics of slowly accreting plate boundaries, J. Geophys. Res. 83 (1978) 3955.

13 V.M. Vainberg, Variational Methods for the Study of Non-linear Operators, (Holden-Day, San Francisco, Calif., 1964) Chapter 2.

14 B.A. Finlayson, The Method of Weighted Residuals and Variational Principles (Academic Press, New York, N.Y., 1972) Chapter 9.

15 C.B. Millikan, On the steady motion of viscous incompressible fluids; with particular reference to a variational principle, Philos. Mag. 7 (1929) 641.

16 P. Glansdorff and I. Prigogine, Thermodynamics of Structure, Stability, and Fluctuations (Wiley, London, 1971).

17 G. Lebon and P. Mathieu, Plane Couette flow on an incompressible non-Newtonian fluid with temperature-dependent viscosity, J. Eng. Math. 6 (1972) 1.

18 D.W. Oldenburg and J.N. Brune, An explanation for the orthogonality of oceanic ridges and transform faults, J. Geophys. Res. 80 (1975) 2575.

19 R.H. Gallagher, J.T. Olden, C. Taylor, and O.C. Zienkiewicz, Finite Elements in Fluids, Vols. 1, 2 (Wiley, London, 1975).